Journal of Statistical Physics, Vol. 101, Nos. 1/2, 2000

Asymptotic Behaviour for Critical Slowing-Down Random Walks[1]

Yves Elskens[2]

Received October 21, 1999; final March 28, 2000

The jump processes $W(t)$ on $[0, \infty[$ with transitions $w \to \alpha w$ at rate bw^β $(0 \leqslant \alpha < 1,\ b > 0,\ \beta > 0)$ are considered. Their moments are shown to decay not faster than algebraically for $t \to \infty$, and an equilibrium probability density is found for a rescaled process $U = (t + \kappa)^{-\beta} W$. A corresponding birth process is discussed.

KEY WORDS: Granular solids; kinetic theory.

1. INTRODUCTION

Random walks with absorbing states occur quite commonly. Here we focus on the asymptotic behaviour of random walks suffering critical slowing down on approaching such a state.[1–3, 6]

Consider a markovian random walk W on $[0, +\infty[$ with continuous time t. The value of W is multiplied at random times by a given factor $0 \leqslant \alpha < 1$, and successive jumps occur independently, with a waiting time between them distributed according to an exponential law with parameter bW^β, where $b > 0$, $\beta > 0$. By introducing a new time $t' = bt$, a new variable $W' = W^\beta$ and a new constant $\alpha' = \alpha^\beta$, one reduces this model to the case $\beta = 1$, $b = 1$ with no loss of generality. A characteristic parameter related to α is the e-folding number of jumps

$$\nu = -1/\ln(\alpha) \tag{1}$$

[1] *Nonlinear Science: Dynamics and Stochasticity*. On the occasion of the 60th birthday of Grégoire Nicolis. Université Libre de Bruxelles, June 30–July 3 (1999).

[2] Equipe turbulence plasma de l'UMR 6633 CNRS—Université de Provence, case 321, Centre de Saint-Jérôme, F-13397 Marseille Cedex 13. E-mail: elskens@newsup.univ-mrs.fr

such that, after cv jumps, W has been reduced by a factor $\alpha^{cv} = e^{-c}$.

Denoting by $f(w, t)$ the probability distribution function of W, the evolution equation for f reads

$$\partial_t f(w, t) = -wf(w, t) + \alpha^{-2} wf(\alpha^{-1} w, t) \qquad \text{for} \quad \alpha > 0 \tag{2}$$

$$\partial_t f(w, t) = -wf(w, t) + \delta(w)\, x_1(t) \qquad \text{for} \quad \alpha = 0 \tag{3}$$

If they exist, the moments $x_k(t) = \langle W^k \rangle = \int_0^\infty w^k f(w, t)\, dw$ satisfy the hierarchy

$$\dot{x}_k = -(1 - \alpha^k)\, x_{k+1} \tag{4}$$

with the dot denoting derivation with respect to time t. We assume that $\mathbb{P}(W(0) > 0) = 1$.

As this random walk can move only towards the origin, we are interested in its asymptotic approach to 0. Equations (2)–(4) indicate that W and t^{-1} are dimensionally homogeneous, and no absolute time scale is available in these equations.

Physically speaking, this random walk appears naturally in the kinetic theory of inelastic systems. Up to dimensional constants, if W is the modulus of the velocity of a particle suffering inelastic collisions (with restitution factor α) with fixed colliders, our process describes the particle's velocity slowing down. For an assembly of such particles, the resulting process is thus inelastic cooling. However, further physical analysis of actual granular systems leads to a more complex process that our simple random walk.[2, 3]

2. EVOLUTION FOR *W*

The linear hierarchy (4) has a straightforward formal solution[1, 6]

$$x_k(t) = \sum_{l=0}^{\infty} \frac{(-t)^l}{l!} \frac{[k+l-1]!}{[k-1]!} x_{k+l}(0) \tag{5}$$

with the q-factorial notation $[k]! = \prod_{n=1}^{k} ((1-\alpha^n)/(1-\alpha))$. Note that $\lim_{\alpha \to 1} [k]! = k!$ and that $0 < \lim_{k \to \infty} (1-\alpha)^k [k]! < 1$ for $0 < \alpha < 1$.

If $x_k(0) < M^k$ for some $M > 0$ uniformly with respect to $k \geqslant 0$, then (5) implies that all moments $x_k(t)$ are entire functions of t, bounded by $M^k e^{(1-\alpha) M |t|}$. Also, if $0 \leqslant f(w, 0) < c \exp(-c' w^\gamma)\ \forall w \in]0, \infty[$ for some $c > 0$, $c' > 0$, $\gamma > 1$, moments x_k ($k \in \mathbb{N}$) are entire functions of time. Conversely, hierarchy (4) is ill-defined if some initial moments are infinite, though

(2) and (3) are well-defined in this case too. These bounds are useful to discuss analytic properties of (5), but they are poor for positive times, as $0 < x_k(t) < x_k(0)$ obviously for $t > 0$.

Consider now the evolution equation (2). By linearity and scale invariance, its general solution reads

$$f(w, t) = \int_0^\infty f(v, 0)\, Q\left(\frac{w}{v}, vt\right) \frac{dv}{v} \tag{6}$$

where Q is the solution of (2) for Dirac initial data $Q(w, 0) = \delta(w-1)$.

For $\alpha = 0$, $Q(w, t) = e^{-t}\delta(w-1) + (1-e^{-t})\,\delta(w)$, so that for $w > 0$, $f(w, t) = e^{-wt} f(w, 0)$. The moments follow readily, $x_k(t) = k!\, t^{-k-1} f(0, 0) + o(t^{-k-1})$, and the surviving fraction of the initial population $\mathbb{P}(W(t) > 0) = t^{-1} f(0, 0) + o(t^{-1})$, if $f(w, 0)$ is smooth near $w = 0$.

For $\alpha > 0$, the representation $Q(w, t) = \sum_{n=0}^{\infty} q_n(t)\, \delta(w - \alpha^n)$ yields coefficients q_n by induction. Indeed, $\dot{q}_0 = -q_0$, $\dot{q}_n = -\alpha^n q_n + \alpha^{n-1} q_{n-1}$, with initial data $q_0(0) = 1$, $q_n(0) = 0$ for $n > 0$. The Laplace transforms $\hat{q}_n(s) = \int_0^\infty q_n(t)\, e^{-st}\, dt$ follow for $\Re(s) > 0$:

$$\hat{q}_0(s) = (1+s)^{-1} \tag{7}$$

$$\hat{q}_n(s) = \frac{\alpha^{n-1}}{s+\alpha^n}\, \hat{q}_{n-1} = \frac{1}{s+1} \prod_{k=1}^{n} \frac{\alpha^{k-1}}{s+\alpha^k} \tag{8}$$

The accumulation of poles $s_n = -\alpha^n$ to the origin reflects the critical slowing down of the process.

The Laplace transforms $\hat{X}_k(s)$ of moments $X_k(t) = \sum_n q_n(t)\, \alpha^{nk}$ have singular expansions for $s \to 0$. In particular, $(1+s)\, \hat{X}_1(s) = 1 + \hat{X}_1(s/\alpha)$ for $s > 0$, which admits the solution $\hat{X}_1(s) = A(s) \ln s + B(s)$ with entire functions $A(s) = \sum_{n=0}^{\infty} a_n s^n$, $B(s) = \sum_{n=0}^{\infty} b_n s^n$ near $s = 0$: $a_n = a_0 \prod_{k=1}^{n} (\alpha^{-k} - 1)^{-1}$, $b_n = (b_{n-1}\alpha^n + a_n \ln \alpha)/(1-\alpha^n)$, $a_0 = v$.

3. RESCALED PROCESS

As the natural time scale for the evolution of W is $1/W$, consider the rescaled variables

$$\tau = \ln(1 + t/\kappa) \tag{9}$$

$$U(\tau) = (t + \kappa)\, W(t) \tag{10}$$

where the characteristic time κ is adapted to the specific initial data f. The moments y_k of U and its probability density h are

$$y_k(t) = (t+\kappa)^k \, x_k(t) \tag{11}$$

$$h(u, \tau) = (t+\kappa) \, f(w, t) \tag{12}$$

and for $0 < \alpha < 1$ obey the equations

$$y'_k = k y_k - (1-\alpha^k) \, y_{k+1} \tag{13}$$

$$y_k(0) = x_k(0) \, \kappa^k \tag{14}$$

$$\partial_\tau h = Lh(u, \tau) + \alpha^{-2} u h(\alpha^{-1} u, \tau) \tag{15}$$

$$h(u, 0) = \kappa^{-1} f(u/\kappa, 0) \tag{16}$$

where $Lh(u, \tau) = -u \partial_u h(u, \tau) - (u+1) \, h(u, \tau)$ and the prime denotes derivation with respect to τ. It is easily seen that U is a stationary Markov process, ergodic on the half line $]0, \infty[$. It jumps down by a factor α with rate u and drifts upwards along exponential characteristics ($u' = u$).

As U is ergodic, one finds a stationary density $h^{\mathrm{eq}}(u)$ and its moments y_k^{eq}:

$$y_{k+1}^{\mathrm{eq}} = y_1^{\mathrm{eq}} \prod_{l=1}^{k} \frac{l}{(1-\alpha^l)} = y_1^{\mathrm{eq}} \frac{k!}{[k]!} (1-\alpha)^{-k} \tag{17}$$

Moreover, as (13)–(14) hold for all $k \in \mathbb{R}$ and $y_0 = 1$ by normalisation, one finds $y_1^{\mathrm{eq}} = \nu = -1/\ln(\alpha)$ in the limit $k \to 0$. Induction for negative k shows that the stationary solution of (13) has also finite moments for all $-\infty < k < +\infty$ (diverging for $|k| \to \infty$).

A series for h^{eq} is found in the form

$$h^{\mathrm{eq}}(u) = \sum_{m=0}^{\infty} \eta_m u^{-1} e^{-u/\alpha^m} \tag{18}$$

with coefficients $\eta_m = -\eta_{m-1} \alpha^{-1} (\alpha^{-m} - 1)^{-1}$ (note that $\eta_m \sim \alpha^{m(m-1)/2}$ for $m \to \infty$). This series converges in the half plane $\Re(u) > 0$, with an essential singularity at $u = 0$. At that point, all derivatives of h^{eq} vanish, in agreement with the finiteness of its moments. Figure 1 displays this stationary density and its moments.

While initial distributions $h(u, 0)$ relax to the stationary density h^{eq}, the distribution of the original variable W approaches[7] this profile, up to the rescaling by $(t+\kappa)$. In particular, Fig. 2 displays the first two moments of the Green function Q and the leading term $(t+\kappa)^{-k} \, y_k^{\mathrm{eq}}$ in its

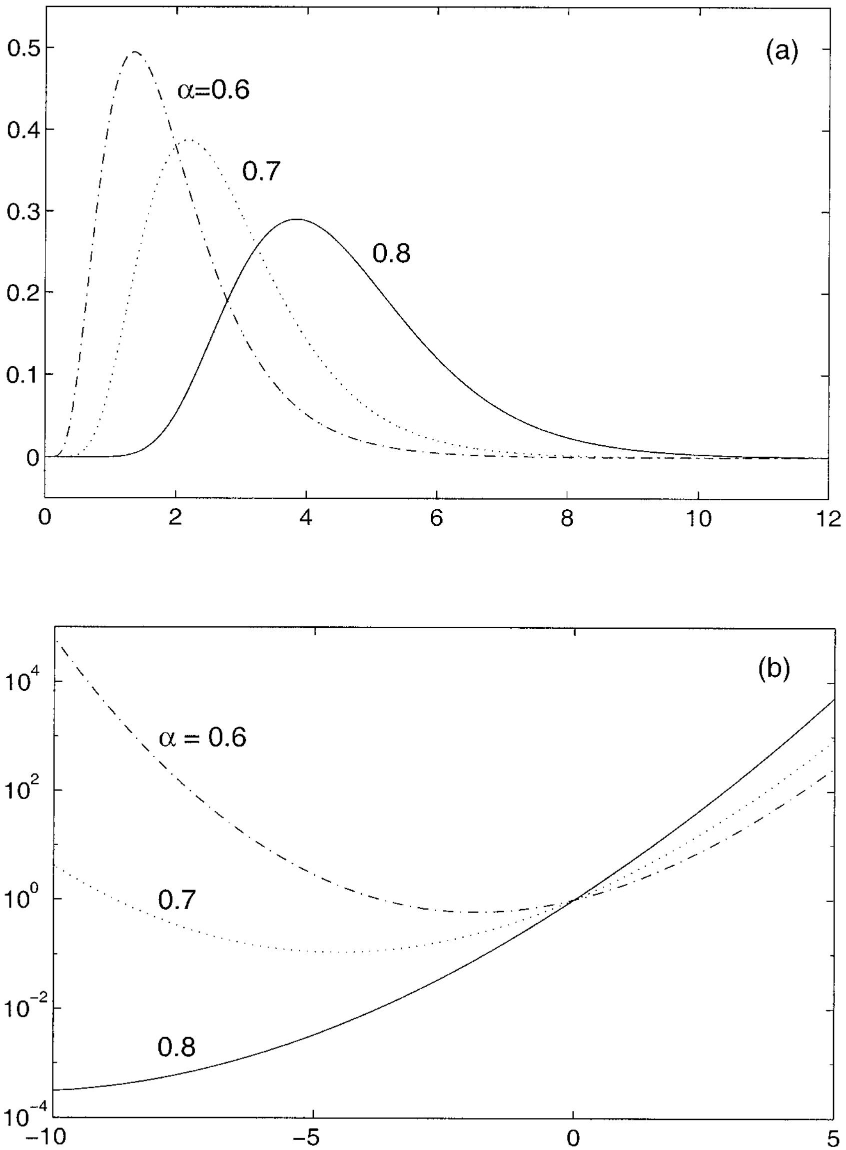

Fig. 1. For $\alpha = 0.8$ (solid line), 0.7 (dots) and 0.6 (dash-dots): (a) invariant density h^{eq} vs u; (b) moments y_k^{eq} vs k (in semi-log scale).

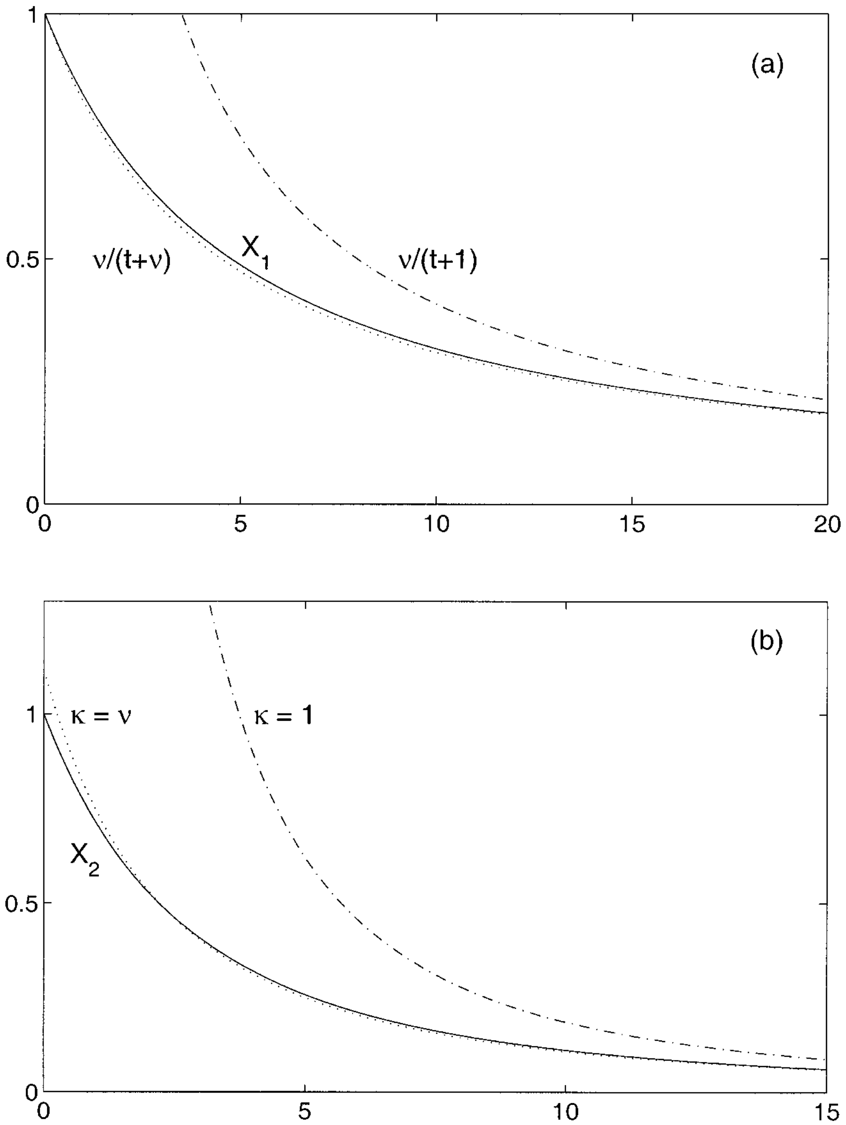

Fig. 2. Expectations $X_k(t)$ for the kernel Q with $\alpha = 0.8$ and (a) $k = 1$, (b) $k = 2$. Solid line: direct sum of series (5) with $x_k(0) = 1$ $\forall k$; thin lines: asymptotically leading approximation $(t+\kappa)^{-k}\, y_k^{\mathrm{eq}}$ with $\kappa = 1$ (dots) and $\kappa = \nu$ (dash-dots).

asymptotic expansion for $t \to \infty$. Two choices for time scale κ are compared: $\kappa = 1$ may seem natural in view of initial data $\mathbb{P}(W(0) = 1) = 1$, but the choice $\kappa = y_1^{\mathrm{eq}}/X_1(0) = v$ has the advantage that the first moment satisfies $y_1(0) = y_1^{\mathrm{eq}}$ and, indeed, the relaxation of $(t+\kappa)\, Q(u/(t+\kappa), t)$ to h^{eq} is almost unnoticed on these lower moments with the choice $\kappa = v$.

4. POPULATION DYNAMICS INTERPRETATION

One may also interpret our process as a birth (only) process for the variable $Z = \ln(\kappa W)/\ln \alpha$ (with the same characteristic time κ as introduced in (9) and (10)): Z increases by unit jumps, with a jump rate $\kappa^{-1}\alpha^{Z}$. This description suggests that Z would describe e.g., a self-inhibiting single-species population growth process, where all present individuals of the species are cooperating to produce one more individual.

It is easily seen that, for the population corresponding to the asymptotic distribution,

$$\langle Z(t) \rangle \approx v \ln(1 + t/\kappa) - v \langle \ln U \rangle^{\mathrm{eq}} \tag{19}$$

where $\langle \ln U \rangle^{\mathrm{eq}} = \lim_{k \to 0} dy_k^{\mathrm{eq}}/dk$. The population grows to infinity, logarithmically in time, as does the solution $z = v \ln(a + t/\kappa) - v \ln v$ (with integration constant a) to the rate equation $\dot{z} = \kappa^{-1}\alpha^{z}$ corresponding to the birth process.

5. CONCLUDING REMARKS

The long-time behaviour of moments x_k in the case $f(w, 0)$ does not vanish on a neighbourhood of $w = 0$ raises interesting questions, as the density h^{eq} is flat at the origin. Relaxation of $(t+\kappa)\, f(u/(t+\kappa), t)$ is likely to be algebraic, and the competition between the scale factor $(t+\kappa)^{-1}$ and the relaxation of h may lead to non-universality in the asymptotics.

One may also consider that the absorption at the origin is a coarse description of the relevant physical processes. In this respect, one may balance the drift towards zero by various simple processes:

1. adding a diffusion term to (2), with reflecting boundary condition $(\partial_w f(0, t) = 0)$;[4]
2. imposing a constant acceleration (with W interpreted as a velocity) between jumps, as in the one-dimensional inelastic Lorentz gas model.[5]

Analytic investigations of these models reveal rich behaviours and varied structures of equilibrium distributions.[4, 5]

ACKNOWLEDGMENTS

It is a pleasure to thank L. Brenig, R. Lambiotte, J. Piasecki, J. M. Salazar Cruz and participants to CEMRACS'99 (CIRM, Luminy) for discussions, and E. Andjel for suggesting to find a rescaled process. The author is pleased to recall his first exposure to random walks in physics and chemical kinetics by G. Nicolis, and that this work initiated from his invitation by G. Nicolis to université libre de Bruxelles last year.

REFERENCES

1. L. Brenig and J. M. Salazar, Exact results for the homogeneous cooling state of an inelastic hard-sphere gas, *J. Plasma Phys.* **59**:639–646 (1998).
2. Y. Elskens, Comment on the paper *Inelastic Granular Gas: Computer Simulations and Kinetic Theory of the Cooling State* by J. M. Salazar and L. Brenig, preprint TP99.14, cond-mat/9911276.
3. M. H. Ernst, Comment II on J. M. Salazar and L. Brenig's contribution *Inelastic Granular Gas: Computer Simulations and Kinetic Theory of the Cooling State*, cond-mat/9911436.
4. R. Lambiotte, private communication.
5. Ph. A. Martin and J. Piasecki, Lorentz's model with dissipative collisions, *Physica A* **265**:19–27 (1999).
6. J. M. Salazar and L. Brenig, Inelastic granular gas: Computer simulations and kinetic theory of the cooling state, *Phys. Rev. E* **59**:2093–2104 (1999).
7. This is clear for initial measures f with compact support in $]0, \infty[$ (bounded away from the origin), such as the Dirac distribution. Asymptotics with $f(0,0) > 0$ or with $f(w, 0)$ decaying too slowly for $w \to \infty$ may be slower.